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Nonlinear evolution of spiral density waves generated by the instability of the shear layer in a rotating compressible fluid

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In a previous paper we considered the nonlinear stability of a cylindrical mixing layer in an incompressible fluid at large Reynolds numbers. Nonlinear evolution results in the formation of vortex structures in the vicinity of the corotation radius r_c. This paper considers the same model but in a compressible fluid. A fundamental difference implied by the presence of compressibility is the possibility of the generation of disturbances which are no longer localized near the shear layer but embrace the entire region. These are acoustic waves generated in the region of corotation resonance and emitted into the periphery. In the $r > r_c$ region lines of equal density are trailing spirals. The nonlinear evolution of such disturbances is determined by redistribution of the mean flow inside the critical layer (CL). It is shown that only two possible types of CL, viscous and unsteady, can be realized here. For both types of these regimes, evolution equations describing the dynamics of a spiral density wave amplitude are obtained and their solutions analysed. It appears that at any values (provided that they are small enough) of initial supercriticality of the flow, an explosive growth of amplitude occurs which continues as long as values comparable with background ones are reached.

1. Introduction

In a previous paper (Shukhman 1989*a*) I considered the nonlinear evolution of a mixing layer in a cylindrical geometry. The fluid was assumed to be incompressible such that disturbances arising as a result of the shear flow instability were localized near the shear layer. Of great interest is the investigation of a mixing layer in a rotating compressible fluid. Such a model can have applications to the problem of the spiral structure of galaxies. According to one of the existing hypotheses, the formation of a spiral structure is attributable to a hydrodynamical instability caused by particular characteristics of rotation curves of galactic gas disks (see, for example, Fridman 1978 and references therein).

Compressibility introduces an essentially new element into the nature of the shear flow instability, i.e. the emission of acoustic waves by the shear layer becomes possible.† Let compressibility be characterized by Mach number $M = R|\Delta\Omega|/c$, where c is the velocity of sound, R is the radius where the shear layer is localized, and

† Strictly speaking, these disturbances can only be called acoustic waves sufficiently far from the critical level. Near it, they have a more complex character and, at small Mach numbers, do not differ greatly from disturbances in an incompressible medium.

$\Delta\Omega = \Omega_1 - \Omega_2$ is the difference between angular velocities in the layer (a more accurate definition of M will be given below). $M = 0$ or $c = \infty$ correspond to the incompressible case. If an extremum is present on the vorticity profile, then the Kelvin–Helmholtz instability arises, and this occurs for any sign of $\Delta\Omega$. A growth of the Mach number, generally speaking, stabilizes this instability; in this case, however, a new type of instability appears, the centrifugal (or radiative) instability. However, it does not occur at all values of the difference $\Delta\Omega$ but only when the internal part rotates faster than the periphery: $\Omega_1 > \Omega_2$ (Morozov 1977, 1979). Specialists in stellar dynamics know an analogue of this instability as the 'instability of circular orbits'. To explain this, we shall consider the limiting case of very large M, or small c, when pressure is negligibly small. Each fluid particle can then be considered independently and the stability of its circular orbit can be investigated relative to a small disturbance. It appears that the circular orbit is unstable if $\kappa^2 < 0$, where κ^2 is the square of the so-called epicyclic frequency: $\kappa^2 = 2\Omega(2\Omega + r\Omega')$, $\Omega(r)$ is the angular velocity of an orbit of radius r, and the prime denotes a derivative in r. In terms of 'cold' ($c = 0$) hydrodynamics this same condition can be formulated as $\zeta < 0$, where $\zeta = (1/r)(r^2\Omega)'$ is the flow vorticity. For $c \neq 0$ ($M \neq \infty$), this criterion is modified but a necessary condition for instability, i.e. the presence of a sufficiently abrupt decrease in angular velocity on part of the profile, remains.

From the formal point of view the new element, in comparison with its incompressible case, implies here a variation of the boundary condition for $r \rightarrow \infty$: instead of the condition of a decreasing disturbance, the radiation condition arises. This factor changes drastically the character of the peculiarity solution for the neutral mode at the point $r = r_c$, where $\Omega(r_c) = \Omega_p$, and Ω_p is the azimuthal phase velocity of the wave pattern. For real boundary conditions (which occur in the incompressible case) and in the presence of only one critical layer on the profile, it is easy to show that the corotation radius $r = r_c$ coincides with the radius at which the vorticity has an extremum, i.e. $\zeta'(r_c) = 0$ (for plane flows $u''(y_c) = 0$). If, however, the boundary conditions are complex ones, then $\zeta'(r_c) \neq 0$. In the first case the Frobenius expansion of the eigenfunction of the neutral mode near $r = r_c$ does not contain peculiarities, i.e. it is a regular one, while in the second case this expansion contains a contribution of the form $(r - r_c)\ln(r - r_c)$ with the proportionality coefficient $\sim\zeta'(r_c)$. Accordingly, the resonance point $r = r_c$ in the first and second cases is customarily called the regular and singular point, respectively. In this paper we want to investigate the nonlinear evolution of a weakly supercritical flow with a singular resonance point, unlike the cases with a regular point that we have investigated before (Churilov & Shukhman 1986, 1986*a*, *b*; Shukhman 1989*a*). We shall demonstrate that, as in the papers just cited, the main nonlinearity is associated with flow redistribution in the critical layer (CL) region, but unlike the situations realized in those papers, here the nonlinearity gives rise to a faster increase of disturbances, thus leading to their explosive increase up to (dimensionless) amplitudes of order unity. Therefore, it is not possible to follow, within the framework of a weakly nonlinear theory, the evolution to its end (to the saturation stage). The rapid increase of disturbances leads to the fact that a nonlinear CL cannot be produced here during the course of the evolution, so that the evolution is proceeding only through regimes with either an unsteady or viscous CL (which, however, with increasing amplitude is also replaced by an unsteady one). The same picture emerges in the case of the evolution of a stratified shear flow (Churilov & Shukhman 1988).

Thus, the objective of this paper is twofold. On the one hand, we want to take the first step towards a nonlinear theory of a compressible differentially rotating fluid, bearing in mind subsequent applications to interpreting laboratory experiments on the modelling of the spiral structure of galaxies (Fridman *et al.* 1985); on the other hand, our intention here is to study a new type of nonlinear evolution equation that arises in this case. As will be shown in the following, a similar equation can be obtained, through certain simplifying modifications, from an equation derived in an earlier paper by Hickernell (1984) for a far simpler model of an incompressible shear flow on the β-plane.

After this paper had been submitted to be considered for publication, a paper by Goldstein & Leib (1989, to be referred to herein as G&L) appeared which had much in common with the present work. It addresses spatially growing instability waves on compressible shear layers between two parallel streams and a nonlinear evolution equation is derived, which actually coincides with that obtained in this paper. However, despite the identity of the equations obtained (except that our equation governs a temporal growth rather than spatial growth), there is a fundamental difference in the critical-layer structure in our paper and in G&L's. G&L examined the situation when the neutral phase velocity is subsonic relative to both streams and in that case – unlike ours – real conditions of disturbance decay instead of radiation conditions are specified as the boundary conditions.† Consequently, the critical level also coincides with the inflexion point. But because G&L also introduced into the model the temperature inhomogeneity, $T_0'(y) \neq 0$, the critical level turned out to coincide not with the usual inflexion point but with a so-called generalized inflexion point, where $(u'/T_0)' = 0$ or $u_c''/u_c' = T_c'/T_c$. Such a modification of the model leads to the structure of eigenfunctions of the neutral mode and, hence, the structure of the inner solution being different from ours. Thus, the pressure disturbance, according to G&L, is regular on the critical level, i.e. does not contain a logarithmic contribution, while for our case this contribution is decisive. Instead, G&L have a different type of singularity: the eigenfunction of the temperature disturbance has a pole $\Theta \sim T_c'/y$ as $y \rightarrow 0$. The main nonlinearity in G&L turns out to be associated just with this singularity. Nevertheless, very surprisingly such an important difference in the critical-layer structure does not lead to a difference in the evolution equations. The nonlinear term in both cases has absolutely the same form, and the difference applies only to the form of the coefficient of the nonlinear term: in G&L it is proportional to T_c' (and disappears for the case with a homogeneous temperature), while in our model, where $T' = 0$, it is proportional to u_c'' (or more exactly, to ζ_c'), which reflects the difference in the origin of the predominant nonlinearity in the two models under consideration.

This paper is organized as follows. In §2 we shall obtain some results using linear theory which will be needed subsequently: we shall find neutral curves corresponding to azimuthal numbers m and shall calculate the instability growth rate in the vicinity of the neutral curves. Section 3 will give the derivation of amplitude equations for the regimes of a viscous and an unsteady CL; and an analysis of their solutions will be made in §4. The results obtained in §4 were also presented in an earlier preprint by this author (Shukhman 1989*b*), having the same title as this paper, and largely coincide with the results obtained almost simultaneously and independently in G&L. The results will be summarized and discussed in §5.

† In this sense G&L's paper is not basically about sound waves, unlike ours, where the radiation of acoustic waves plays a fundamental role.

2. Linear theory

We take a two-dimensional system of the Navier–Stokes equation and the continuity equation,

$$\left.\begin{aligned}\rho\left[\frac{\partial \boldsymbol{v}}{\partial t}+(\boldsymbol{v}\nabla)\,\boldsymbol{v}\right] &= -\nabla p+\eta(\Delta \boldsymbol{v}+\tfrac{1}{3}(\nabla(\nabla\cdot \boldsymbol{v}))+\boldsymbol{F}_g,\\ \frac{\partial \rho}{\partial t}+\nabla\cdot(\rho \boldsymbol{v}) &= 0,\end{aligned}\right\} \tag{2.1}$$

as the initial system of equations. We choose a flow with $v_r = 0$ and $v_\varphi = r\Omega(r)$, where

$$\Omega(r) = \tfrac{1}{2}\Omega_1[1-\tanh(\ln(r/R)/D)] \tag{2.2}$$

as the undisturbed flow. Here v_r and v_φ are the radial and azimuthal velocities, respectively, and $\Omega(r)$ is the angular velocity. It is evident that when $r\to 0$, $\Omega\to\Omega_1$ and the angular velocity on the periphery Ω_2 is zero such that $\Delta\Omega\equiv\Omega_1-\Omega_2=\Omega_1$. The undisturbed pressure p_{00} and density ρ_{00} will be taken independent of both the radius as well as the sound velocity c.† Viscosity η is taken constant. It is supposed that the centrifugal force is balanced by some external force $\boldsymbol{F}_g$. In the case of galaxies this is the gravitational force produced by a stellar subsystem (see e.g. Chandrasekhar 1961, and Fridman & Polyachenko 1984).

In what follows, we shall be using dimensionless variables, by putting $R_0 = 1$, $\Delta\Omega = \Omega_1 = 1$, and $\rho_{00} = 1$. We also introduce the Mach number $M = R_0\Delta\Omega/c\equiv c^{-1}$ and the inverse of the Reynolds number $\nu = \eta/(\rho_{00}R_0^2\Delta\Omega)\equiv\eta$ which will be assumed small such that it need be taken into account only near the critical level. By linearizing the system (2.1) written in cylindrical coordinates and omitting the viscous term, we reduce it to one equation for the density perturbation:

$$\begin{aligned}\frac{1}{r}\hat{L}(\Omega_p,M,D)\,\rho_1 &\equiv \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{r}{\Delta}\frac{\partial\rho_1}{\partial r}\right)-\frac{m^2}{r^2}\frac{\rho_1}{\Delta}\\ &\quad-\frac{2\rho_1}{r}\left[\frac{m^2\Omega'(\Omega_p+\Omega)}{\Delta^2}+\frac{2\Omega^2}{\Delta^2}\left(\frac{\kappa^2}{2\Omega}\right)'\frac{1}{\Omega_p-\Omega}\right]+M^2\rho_1 = 0.\end{aligned} \tag{2.3}$$

Here $\Delta = \Delta(r,\Omega_p) = m^2(\Omega_p-\Omega)^2-\kappa^2, \kappa^2 = 2\Omega\zeta$. All disturbances are chosen to be of the form $\sim\exp[im(\varphi-\Omega_p t)]$.

Precisely the same equation occurs for pressure disturbance owing to the simple connection $p_1 = M^{-2}\rho_1$. The boundary conditions for (2.3) are written as

$$\rho_1\to 0 \quad\text{when}\quad r\to 0;\quad \rho_1'-ik\rho_1\to 0\quad\text{when}\quad r\to\infty, \tag{2.4}$$

where $k = m\Omega_p c^{-1} = Mm\Omega_p$ is a radial wavenumber of an outgoing acoustic wave.

2.1. *Analysis of neutral modes*

For each m, equation (2.3) involves three parameters: M, D and Ω_p, where Ω_p, generally speaking, is complex. By solving (2.3) with the boundary conditions (2.4), we obtain

$$\Omega_p = \operatorname{Re}\Omega_p(M,D)+\mathrm{i}\operatorname{Im}\Omega_p(M,D).$$

† In this case we can take, without loss of generality, the simplified equation $p = \rho c^2$ as the energy equation to complete the system (2.1).

The neutral curve is defined by the condition $\operatorname{Im} \Omega_p(M,D) = 0$ which gives the relationship $D = D_m(M)$. In this case $\Omega_p = \Omega_p(M)$. (The case $m = 0$ must be considered separately.)

Let us discuss the properties of the neutral modes. In the limiting case, corresponding to $M = 0$, we have (Shukhman 1989*a*):

$$\left.\begin{aligned} D_m(M=0) &= \frac{1}{m}, \quad \Omega_p(M=0) = \frac{1}{2}\left(1-\frac{1}{m}\right), \quad m = 1,2,\ldots, \\ \rho_m(r,M=0) &= \cosh\left(\frac{y}{D}\right)\left\{1-\frac{1}{2}\left(\frac{1}{D}+\tanh\frac{y}{D}\right)\right\}, \quad y = \ln r. \end{aligned}\right\} \tag{2.5}$$

It is also easy to find the equation of the neutral curve for $M \ll 1$:

$$\left.\begin{aligned} D_m(M) &= D_m(0)+\delta D_m, \quad \Omega_p(M) = \Omega_p(0)+\delta\Omega_p, \quad \delta\Omega_p = -\tfrac{1}{2}\delta D_m, \\ \delta D_m &= \frac{1}{24\sin(\pi/m)}\frac{1}{m^2}\left(1-\frac{1}{m}\right)\left(\frac{2}{m^3}-\frac{10}{m^2}+\frac{9}{m}-3\right)M^2. \end{aligned}\right\} \tag{2.6}$$

Formula (2.6) shows that $\delta D_m < 0$. This corresponds to the intuitive idea – which rests on results reported by Landau (1944) that unstable two-dimensional disturbances of a plane tangential discontinuity are stabilized with increasing Mach number – of the stabilizing role of compressibility as long as M is small: the range of unstable values of the shear width D becomes narrower ($0 \leqslant D < D_m$).

Equation (2.3), with real Ω_p, contains one singular point, a corotation resonance $r = r_c$, which must be understood in the sense of the Lin indentation rule, i.e. in our case from above because $\Omega'(r_c) < 0$. (One can show that at points of the so-called Lindblad resonance $r = r_L$, where $\Delta(r_L) = 0$ or $\Omega(r_L)-\Omega_p = \pm(1/m)\,\kappa(r_L)$, the solution (2.3) does not have any singularities.) The presence of such a point leads to the fact that the real equation (2.3) has a complex solution. It is because of this that it becomes possible to satisfy the complex boundary condition (2.4). Let us discuss this point in greater detail.

When $r \to \infty$, (2.3) has an asymptotic representation $\rho_1 = (\rho_\infty/r^{\frac{1}{2}})\,e^{ikr}$. We denote $\rho_1(r_c) = \rho_c$. Upon multiplying (2.3) by $\bar{\rho}_1$, where the overbar indicates complex conjugacy, and subtracting from the expression obtained the complex conjugate one, as well as assuming that $(\Omega_p-\Omega)^{-1} \to (\Omega_p-\Omega+i\delta)$, $\delta \to +0$, we obtain a radiative identity relating the amplitude of the outgoing wave ρ_∞ to the value of the eigenfunction and to the flow parameters when $r = r_c$:

$$\frac{M}{m\Omega_p}|\rho_\infty|^2 = -\left(\frac{2\Omega}{\kappa^2}\right)_c \frac{(\kappa^2/2\Omega)'_c}{|\Omega'|_c}|\rho_c|^2. \tag{2.7}$$

It becomes understandable from this identity that neutral spiral waves become possible only in the presence of corotation, and $(\kappa^2/2\Omega)'_c \neq 0$ must hold. This also constitutes the essence of the 'anti-spiral theorem' of the theory of the spiral structure of galaxies (Lynden-Bell & Ostriker 1967).

From (2.7) it follows that the corotation lies in the region of those values of r where $(\kappa^2/2\Omega)'_c \sim \zeta'_c < 0$. This establishes the range of possible values of Ω_p corresponding to the neutral curve (see figure 1): $1 \geqslant \Omega_p \geqslant \frac{1}{2}(1-D) = \Omega_p(M=0)$. With increasing M, Ω_p increases compared with Ω_p in the incompressible case, the corotation radius is displaced towards the rotation axis.

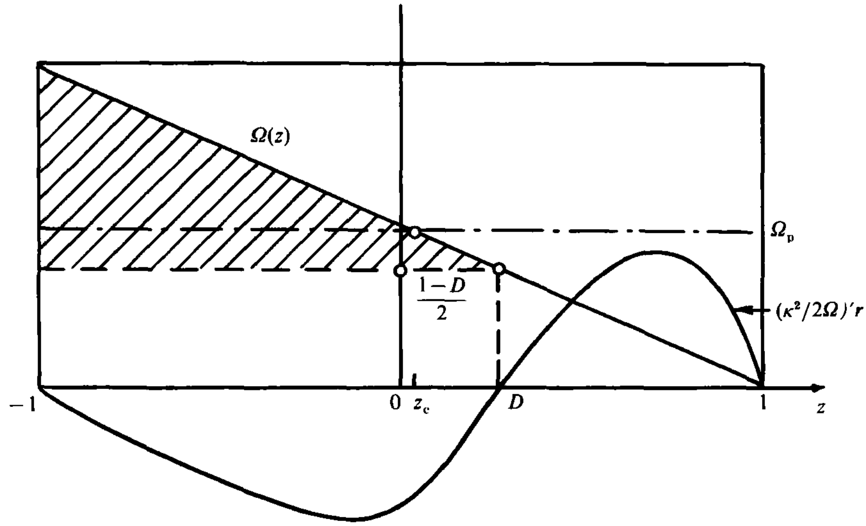

FIGURE 1. An explanation of the method of seeking the range of possible values of Ω_p for the neutral curves: $\frac{1}{2}(1-D) < \Omega_p < 1$. Here $z = \tanh(\ln(r)/D)$, $\Omega = \frac{1}{2}(1-z)$, and $r(\kappa^2/2\Omega) = (1-z^2)(z/D-1)/D)$. The value $z = z_c$ corresponds to the corotation radius. According to the radiative identity, the corotation is located in a region where $(\kappa^2/2\Omega)' < 0$ (i.e. when $z < D$).

The form of the density variation ρ can be described qualitatively in the following manner. In a frame of reference rotating with angular velocity Ω_p, the mode with a given m is, where $r < r_c$, a disturbance of non-spiral form such as a standing wave (i.e. the eigenfunction is real with an appropriate choice of the amplitude) with m-maxima and m-minima azimuthally – 'cart wheel', and when $r > r_c$ it represents a trailing m-arm spiral wave. (At a sufficient distance from the shear region the asymptotic behaviour of the radial part ρ_1 has the form $\rho_1 \sim J_m(Kr)$ inside and $\rho_1 \sim H_m^{(1)}(Kr)$ outside the corotation radius; $K = M((\Omega_p-1)^2-4)^{\frac{1}{2}}$. J_m and $H_m^{(1)}$ are the Bessel and Hankel functions, respectively and their appearance is associated with the fact that far from the shear region (2.3) is reduced to the Bessel equation, owing to the disappearance (to an exponential accuracy) of all gradients Ω', κ' etc. In a laboratory system the wave has a radial velocity directed outwards and equal to the sound velocity c.

The mode $m = 1$ is worthy of a special discussion. In the incompressible case $M = 0$ it is a neutral one for any D (Shukhman 1989*a*). When $M \neq 0$, this degeneracy is cancelled. Analysis shows that when $M \ll 1$ the mode is unstable at $0 \leqslant D < 1$ (and more exactly, when $0 \leqslant D < 1-\frac{1}{12}M^2$), with the growth rate

$$\gamma_L = m \operatorname{Im} \Omega_p = \{\tfrac{1}{8}M^2 F(D)\}^{1/(2-D)} \cos\frac{\pi D}{2(2-D)}, \tag{2.8}$$

where
$$F(D) = \frac{2\pi D}{\sin(2\pi D)} \frac{\pi D}{\sin(\pi D)} (1-D)^2(1-2D)(1-\tfrac{2}{3}D), \tag{2.9}$$

so that for this mode the part of the D-axis from 0 to 1 is also the boundary of stability. The derivation of (2.8) and (2.9) is given in detail in the Appendix.

The neutral curves $D = D_m(M)$ and $\Omega_p = \Omega_p(M)$ for $m = 1$, 2, 3, and 4 were determined numerically in the range $0 < M \leqslant 10$. The results are given in figure 2.

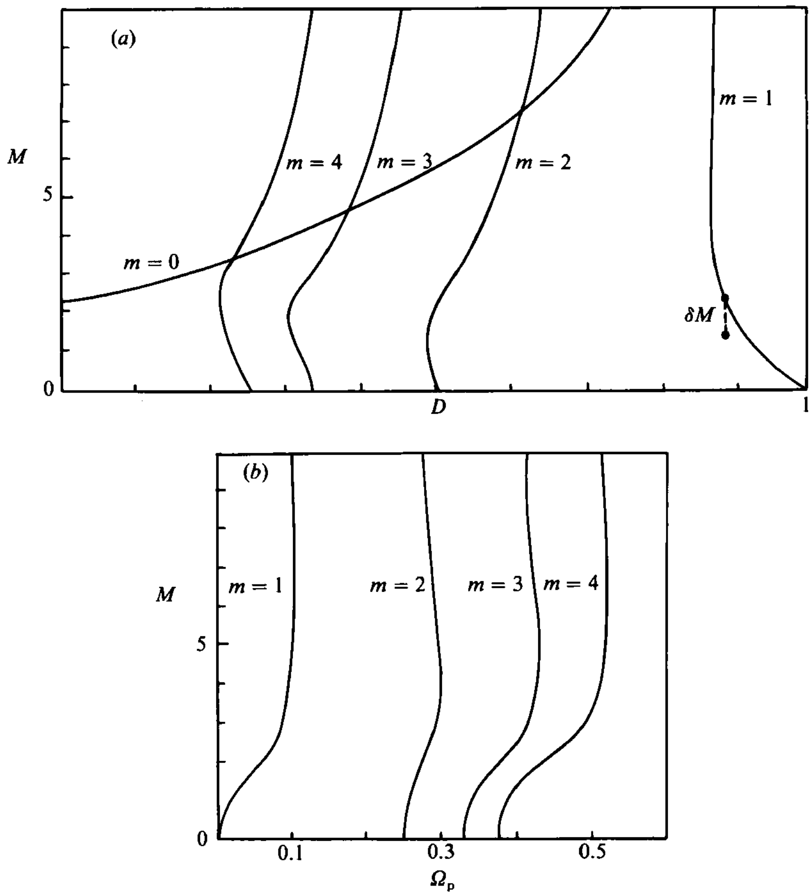

FIGURE 2. (a) Neutral curves $D = D_m(M)$ for different m. The instability region of each mode lies to the left of the corresponding curve. For $m = 1$, a segment of the abscissa axis $0 < D \leqslant 1$ also belongs to the neutral curve. For the $m = 0$ mode, the value of M for the case of a tangential discontinuity $D = 0$ is determined from the relation $MI_1(2M)/I_0(2M) = 2$ to be 2.269. ($I(z)$ is Bessel's function of the imaginary argument.) (b) As (a) but for the azimuthal phase velocity Ω_p.

2.2. *The instability*

The neutral curves are, at the same time, the boundary of stability. It is easy to see that for each mode m the instability region lies to the left of the corresponding neutral curve. This is also confirmed by direct calculations. Moving away from the neutral curve in M (at a fixed D) or in D (at a fixed M) and assuming $\Omega_p \rightarrow \Omega_p + \delta\Omega_p$ we obtain using the well-known procedure of perturbation theory

$$I_t\,\delta\Omega_p + I_M\,\delta M \equiv \int_C Q(r)\,\rho_m(r)\,\mathrm{d}r = 0, \tag{2.10}$$

where (see also (3.15))

$$I_t = -\int_C \mathrm{d}r\left\{\rho_m \frac{\partial\hat{L}}{\partial\Omega_p}\rho_m + \left(\frac{r}{\Delta}\rho_m^2\right)' \frac{\mathrm{i}m^2\Omega_p M^2}{k}\right\}, \tag{2.11}$$

$$I_M = -\int_C \mathrm{d}r\left\{\rho_m \frac{\partial\hat{L}}{\partial M}\rho_m + \left(\frac{r}{\Delta}\rho_m^2\right)' \frac{\mathrm{i}\Delta_\infty M}{k}\right\} \tag{2.12}$$

or, in explicit form

$$I_t = -2m^2 \int_C \mathrm{d}r \left\{ \frac{M^2 r}{\Delta}(\Omega_\mathrm{p}-\Omega)\rho_m^2 - \left(\frac{r}{\Delta}\rho_m^2\right)' \frac{M^2\Omega_\mathrm{p}}{2\mathrm{i}k} + \rho_m^2 \left[-\frac{\Omega'}{\Delta^2} \right.\right.$$
$$\left.\left. + \frac{2\Omega' m^2(\Omega_\mathrm{p}^2-\Omega^2)}{\Delta^3} + \left(\frac{\kappa^2}{2\Omega}\right)' \frac{2\Omega^2}{\Delta^2}\left(\frac{2}{\Delta}+\frac{1}{m^2(\Omega_\mathrm{p}-\Omega)^2}\right)\right] - \frac{r}{\Delta}\rho_m \rho_m' \left(\frac{\Omega_\mathrm{p}-\Omega}{\Delta}\right)' \right\}, \quad (2.13)$$

$$I_M = -\int_C \mathrm{d}r \left\{ 2M^2 r \rho_m^2 - \left(\frac{r}{\Delta}\rho_m^2\right)' \frac{\Delta_\infty M}{\mathrm{i}k} \right\}. \quad (2.14)$$

Here $\Delta_\infty = \Delta(r=\infty) = m^2\Omega_\mathrm{p}^2$. Contour C implies integration from 0 to ∞ with indenting of the singular point $r = r_\mathrm{c}$ from above, $\rho_m(r)$ being the eigenfunction of the neutral mode.

The dispersion equation (2.10) can also be written as a linear evolution equation (if we put $\delta\Omega_\mathrm{p} A \rightarrow (\mathrm{i}/m)(\mathrm{d}A/\mathrm{d}t)$):

$$\frac{\mathrm{d}A}{\mathrm{d}t} - (\gamma_\mathrm{L} - \mathrm{i}m\Delta\Omega_\mathrm{p})A = 0, \quad (2.15)$$

where

$$\gamma_\mathrm{L} = -\operatorname{Im}(I_M/I_t)\, m\delta M, \quad (2.16)$$

$$\Delta\Omega_\mathrm{p} = -\operatorname{Re}(I_M/I_t)\,\delta M. \quad (2.17)$$

Here γ_L and $\Delta\Omega_\mathrm{p}$ are the growth rate and a correction to the phase velocity, and $A(t)$ is the wave amplitude which is defined by the relationship $\rho_m(r) = A(t)\varphi_a(r)$.

Some words about the $m = 0$ mode are in order. This mode does not have a critical level. The marginal stability is defined by the relationship $\omega^2 = 0$, where the disturbance is chosen in the form $\sim\exp(-\mathrm{i}\omega t)$. The eigenfunction of the neutral mode for ρ describes a ring localized near the shear layer. The neutral curve is given in figure 2. The mode is unstable above the neutral curve. We shall not comment on this mode in detail because it is of no special interest within the context of this paper.

We want to construct a weakly nonlinear instability theory for finite values of M. It is clear that with finite M such a theory is completely correct only in relation to the $m = 1$ mode near its neutral curve. Indeed, near the marginal stability of any other mode $m = m_0 > 1$, modes with $m < m_0$ have a finite (not small) growth rate, and their rapid increase will inevitably distort the picture obtained of the evolution of the mode $m = m_0$. Therefore, although the evolution equations obtained in this paper are formally applicable for any m, we should keep in mind that the true weakly supercritical regime is possible for $m = 1$ only.

In summarizing this Section, we shall give the Frobenius expansion of the eigenfunction of the neutral mode. Assuming $z = r - r_\mathrm{c}$, we write

$$\varphi_a^\pm(r) = 1 + q_1 z^2 + q(\ln|z| + B^\pm)(z + q_2 z^2 + \ldots) + \ldots, \quad (2.18)$$

where

$$\left.\begin{aligned} q &= 2m^2\left(\frac{\kappa^2}{r}\right)_\mathrm{c}\frac{f_\mathrm{c}}{\Omega_\mathrm{c}'} = \frac{1}{r_\mathrm{c}}\left(\frac{2\Omega}{\kappa^2}\right)_\mathrm{c}\left(\frac{2\Omega}{\Omega'}\right)_\mathrm{c}\left(\frac{\kappa^2}{2\Omega}\right)_\mathrm{c} > 0, \quad f(r) = \frac{1}{2m^2}\frac{4\Omega^2}{\Delta^2}\left(\frac{\kappa^2}{2\Omega}\right)', \\ q_1 &= \frac{1}{2}\left[\tfrac{3}{2}q^2 - \tfrac{1}{2}q\left(\frac{\kappa^2}{r}\right)_\mathrm{c}\left(\frac{r}{\kappa^2}\right)_\mathrm{c}' + \frac{m^2}{r_\mathrm{c}^2} + q\left(\frac{f_\mathrm{c}'}{f_\mathrm{c}} - \frac{\Omega_\mathrm{c}''}{2\Omega_\mathrm{c}'}\right) + \frac{1}{r_\mathrm{c}}M^2\kappa_\mathrm{c}^2 - \frac{4m^2}{\kappa_\mathrm{c}^2}\frac{\Omega_\mathrm{c}\Omega_\mathrm{c}'}{r_\mathrm{c}}\right], \\ q_2 &= \frac{1}{2}\left[q - \left(\frac{\kappa^2}{r}\right)_\mathrm{c}\left(\frac{r}{\kappa^2}\right)_\mathrm{c}'\right]. \end{aligned}\right\} \quad (2.19)$$

The Lin indentation rule gives for the jump B:

$$B^{+}-B^{-}=-\mathrm{i}\pi \tag{2.20}$$

(see §4). It must also be stressed that the presence of a logarithmic term, i.e. the singularity of the resonance point $r=r_c$, is associated with the difference from zero of the derivative of vorticity on the CL: $(\kappa^2/2\Omega)'_c \neq 0$. Using the notation of (2.19) the linear evolution equation (2.10) can be written in a form which will be more appropriate in the subsequent use of the method of matching asymptotic expansions (see (3.47))

$$\mathrm{FP}\int_0^\infty Q\varphi_a\,\mathrm{d}r+\frac{f_c}{\Omega'_c}\left[\left(\frac{f'_c}{f_c}-\frac{\Omega''_c}{\Omega'_c}\right)\mathrm{i}\pi+q(2\mathrm{i}\pi B^{-}+\pi^2)\right]2m^2\delta\Omega_p\equiv\int_C Q\varphi_a\,\mathrm{d}r=0, \tag{2.21}$$

where $\mathrm{FP}\int_0^\infty(\ldots)\,\mathrm{d}r$ denotes the finite part of the integral with the integrand containing the non-integrable singularity.

3. The derivation of the evolution equation

3.1. *Preliminary analysis*

As in other problems dealing with the critical layer, the main nonlinearity in the evolution equations is determined by processes occurring within it (internal nonlinearity). This makes it possible to obtain nonlinear evolution equations in which the nonlinear terms have a universal character and are independent of the flow structure as a whole. Analysis of the situation treated in this paper, whose distinctive feature is the singularity of the resonance point $r=r_c$ (in the sense characterized above), shows that symbolically the evolution equation can be written as

$$\frac{\mathrm{d}A}{\mathrm{d}t}=\gamma_L A+A\sum_{n-1}^{\infty}C_n\left(\frac{A^2}{l^4}\right)^{2n}. \tag{3.1}$$

(In the linear part of the equation we omit the term with $\mathrm{i}m\Delta\Omega_p$ which is present in (2.15) because it is readily eliminated by redefining the amplitude.) The expansion of the nonlinear term is done in A^2/l^4, where l is the CL scale. As is well known, this scale coincides with the largest of the three scales: $l_\nu \sim \nu^{\frac{1}{3}}$, $l_t \sim \gamma$, and $l_N \sim A^{\frac{1}{2}}$, where l_ν, l_t, and l_N are the viscous, unsteady and nonlinear scales, respectively, $\gamma = A^{-1}\,\mathrm{d}A/\mathrm{d}t$, and ν is inverse of the Reynolds number. In the case $l=l_t$, the notation l^{-1} corresponds to the integral operator $\hat{\gamma}^{-1}$ which denotes, roughly speaking, integration in time and coincides, in order of magnitude, with γ. The coefficients C_n are proportional to $(\kappa^2/2\Omega)'_c$.

In other words, the expansion parameter is

$$\lambda=\left(\frac{l_N}{l}\right)^4. \tag{3.2}$$

As long as the amplitude is sufficiently small and the CL remains linear, i.e. as long as $l \gg l_N$, where l is one of two linear scales $l=l_\nu$ or $l=l_t$, this parameter is, indeed, small. Taking this into account, let us consider what the evolution equation must be like for different CL regimes and what CL regimes are, generally speaking, possible here. We use the amplitude–supercriticality diagram (figure 3). As usual, we want to consider the evolution of a disturbance with a very small initial amplitude, i.e. starting from the lower part of the diagram. Two cases should be distinguished here: (i) $\gamma_L < \nu^{\frac{1}{3}}$, and (ii) $\gamma_L > \nu^{\frac{1}{3}}$. In the former the initial CL regime is a viscous one, and in the latter it is an unsteady one. Let us consider these cases in more detail.

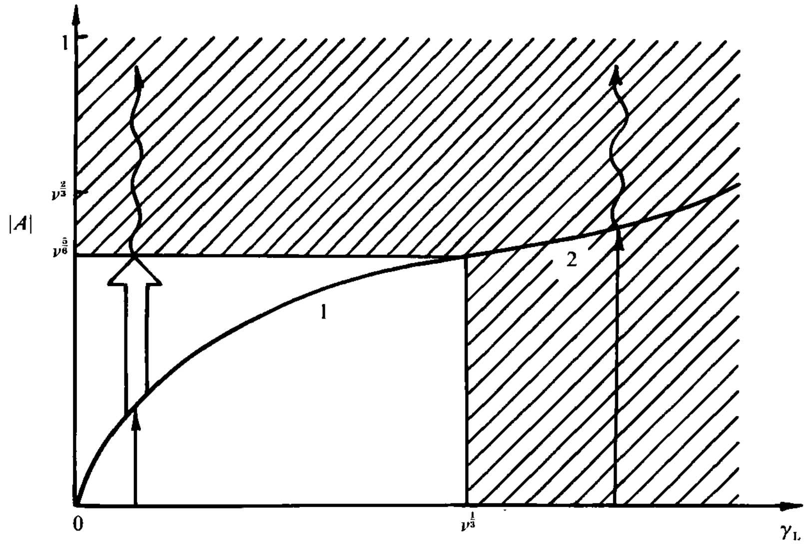

FIGURE 3. The amplitude–supercriticality diagram. The unsteady CL regime is shaded, and the viscous CL regime is unshaded. Curve 1, nonlinearity threshold for a viscous CL: $A \sim A_1 \approx (\gamma_L \nu^{\frac{4}{3}})^{\frac{1}{2}}$; curve 2, nonlinearity threshold for an unsteady CL: $A \sim A_2 \approx \gamma_L^{\frac{5}{2}}$. Vertical arrows indicate different stages of evolution: $\rightarrow$, $|A| \sim \exp(\gamma_L t)$; $\Rightarrow$, $|A| \sim (t_0-t)^{-\frac{1}{2}}$; $\rightsquigarrow$, $|A| \sim (t-t)^{-\frac{5}{2}}$.

(i) $\gamma_L < \nu^{\frac{1}{3}}$. We have $l = l_\nu = \nu^{\frac{1}{3}} \gg l_t, l_N$, and our symbolic equation takes the form:

$$\frac{dA}{dt} = \gamma_L A + C_1 \frac{A^3}{\nu^{\frac{4}{3}}}. \tag{3.3}$$

The level of competitive nonlinearity (or the threshold of nonlinearity, i.e. the value of the amplitude for which the nonlinear term is comparable with the linear one),

$$A_1 \sim (\gamma_L \nu^{\frac{4}{3}})^{\frac{1}{2}}, \tag{3.4}$$

is shown on the diagram by curve 1. As will be apparent below, the sign of the nonlinear term is such that the nonlinearity exerts a destabilizing action. Therefore, the question arises as to what CL regime will occur when the amplitude exceeds this level, i.e. when $A > A_1$.

When $A > A_1$, we have $\gamma \sim A^2\nu^{-\frac{4}{3}}$ and $l_t \sim \gamma \sim A^2\nu^{-\frac{4}{3}}$. On comparing l_t, l_ν and l_N, we obtain that $l_\nu > l_t, l_N$ when $A < \nu^{\frac{5}{6}}$. Thus, when $A < \nu^{\frac{5}{6}}$ and $\gamma_L < \nu^{\frac{1}{3}}$, the CL regime remains a viscous one, and the equation does, indeed, have the form (3.3).

Now let $A > \nu^{\frac{5}{6}}$ (but $\nu^{\frac{1}{3}}$ is again larger than γ_L). We have $l_t > l_\nu$ and, consequently, $l_t \sim \gamma$ should be taken as l in (3.1). In symbolic form the evolution equation assumes the form

$$\frac{dA}{dt} = \gamma_L A + \hat{\gamma}^{-4} A^3. \tag{3.5}$$

Since in this case $\gamma \sim A^{\frac{2}{5}} \gg l_N \sim A^{\frac{1}{2}}$, the CL will be of unsteady rather than nonlinear type. Moreover, noting that in (3.1) the expansion proceeds virtually in the quantity $(l_N/l)^4$, we conclude that it is not necessary to take account of higher-than-cubic terms because the inequality $l \gg l_N$ remains valid throughout the entire evolution up to amplitudes $A \sim O(1)$.

(ii) $\gamma_L > \nu^{\frac{1}{3}}$. In this case we at once have equation (3.5) with the level of competitive nonlinearity

$$A_2 \sim \gamma_L^{\frac{5}{2}} \tag{3.6}$$

shown on the diagram by curve 2 which is matched with curve 1 on the boundary $\gamma_{\mathrm{L}} \sim \nu^{\frac{1}{3}}$ (with amplitude $A \sim \nu^{\frac{5}{6}}$).

Thus, qualitative analysis shows that on the (A, γ_{L})-diagram the viscous CL region is bounded by straight lines $A = \nu^{\frac{5}{6}}$, and $\gamma_{\mathrm{L}} = \nu^{\frac{1}{3}}$ and the remaining region (but where, of course, $A \ll 1$ and $\gamma_{\mathrm{L}} \ll 1$) corresponds to an unsteady CL. The nonlinear CL cannot be realized in the course of the evolution (at least, within the framework of the weakly nonlinear theory, i.e. as long as $A \ll 1$).

Our problem is that of writing the explicit form of the nonlinear evolution equations in regimes with a viscous or an unsteady CL instead of their symbolic form (3.3) and (3.5) and, using them, studying the evolutionary behaviour of the disturbance.

3.2. *The evolution equation in the regime with an unsteady CL*

As usual, it is necessary to solve separately the outer and inner (with respect to the CL) problems and to match the asymptotic expansions of the solutions.

3.2.1. *Scaling*

According to §3.1, we put

$$\frac{\partial}{\partial t} = \mu \frac{\partial}{\partial \tau} - \Omega_{\mathrm{p}} \frac{\partial}{\partial \varphi}, \quad \rho = 1 + \epsilon \delta \rho, \quad \delta M = \mu M_1, \tag{3.7}$$

where ϵ and μ are the small parameters that characterize the disturbance amplitude and the growth rate of disturbances ($\gamma \sim O(\mu)$). The parameter μ also defines simultaneously the scale of the unsteady CL. In addition,

$$\mu = \epsilon^{\frac{2}{5}}.$$

We expand the disturbed quantities in harmonics:

$$\delta \rho = \sum_{l=0}^{\infty} \rho_l(\tau, r) \exp\left[\mathrm{i}m(\varphi - \Omega_{\mathrm{p}} t)\right]. \tag{3.8}$$

3.2.2. *The outer problem*

For the outer problem, we actually need only the expansion of the fundamental ($l = 1$) harmonic. Assuming in (2.3)

$$\left.\begin{aligned} \Omega_{\mathrm{p}} \rightarrow \Omega_{\mathrm{p}} + \frac{\mathrm{i}\mu}{m} \frac{\partial}{\partial \tau}, \quad M \rightarrow M + \mu M_1, \quad \rho_1 = \rho^{(0)} + \mu \rho^{(1)} + \ldots, \\ \frac{\partial}{\partial r} \rightarrow \frac{\partial}{\partial r} + \mu \frac{\partial}{\partial R}, \end{aligned}\right\} \tag{3.9}$$

where the last relationship is need to take into account accurately the region $r \rightarrow \infty$, we perform two steps of the iteration process in powers of μ.

We have

$$\hat{L}^{(0)} \rho^{(0)} = 0, \tag{3.10}$$

whence we obtain

$$\rho^{(0)} = A(\tau, R) \varphi_a^{\pm}(r). \tag{3.11}$$

The next step is

$$\hat{L}^{(0)} \rho^{(1)} = Q^{\pm}(r, R), \tag{3.12}$$

where

$$Q^{\pm}(r, R) = -\frac{\mathrm{i}}{m} \frac{\partial A}{\partial \tau} \frac{\partial \hat{L}}{\partial \Omega_{\mathrm{p}}} \varphi_a^{\pm} - M_1 A \frac{\partial \hat{L}}{\partial M} \varphi_a^{\pm} - \left(\frac{r}{\Delta} \varphi_a^{\pm\prime} + \left(\frac{r}{\Delta} \varphi_a^{\pm}\right)'\right) \frac{\partial A}{\partial R}. \tag{3.13}$$

Taking into account that $\varphi_a^+ \sim r^{-\frac{1}{2}} e^{ikr}$ when $r \to \infty$, from the condition for the absence of secular terms in $\rho^{(1)}$ (in other words, from the condition that $\int^\infty Q^+(r)\varphi_a^+(r)\,dr$ converges on the upper limit), we obtain

$$\frac{\partial A}{\partial R} = \left(\frac{i}{m}\frac{\partial}{\partial \tau} + \Omega_p M_1\right) A, \tag{3.14}$$

so that for $Q(r)$ we get

$$\begin{aligned}\varphi_a^\pm(r)\, Q^\pm(r) &= -2MM_1 A\left\{r\varphi_a^{\pm 2} - \frac{\Delta_\infty}{2ik}\left(\frac{r}{\Delta}\varphi_a^{\pm 2}\right)'\right\} \\ &-2im\frac{\partial A}{\partial \tau}\left\{rM^2\frac{\Omega_p - \Omega}{\Delta}\varphi_a^{+2} - \frac{M}{2ik}\Omega_p\left(\frac{r}{\Delta}\varphi_a^{\pm 2}\right)' - \varphi_a^{\pm 2}\left[-\frac{\Omega'}{\Delta^2} + \frac{2\Omega' m^2(\Omega_p^2 - \Omega^2)}{\Delta^3}\right.\right. \\ &\left.\left. + \left(\frac{\kappa^2}{2\Omega}\right)'\frac{2\Omega^2}{\Delta^2}\times\left(\frac{2}{\Delta} + \frac{1}{m^2(\Omega_p - \Omega)^2}\right)\right] - \frac{r}{2\Delta}(\varphi_a^{\pm 2})'\left(\frac{\Omega_p - \Omega}{\Delta}\right)'\right\}.\end{aligned} \tag{3.15}$$

From (3.12) we find $\rho^{(1)}$:

$$\rho^{(1)\pm}(r) = \int^r dx[\varphi_a^\pm(r)\varphi_b^\pm(x) - \varphi_a^\pm(x)\varphi_b^\pm(r)]\,Q^\pm(x) + a^\pm\varphi_a^\pm(r) + b^\pm\varphi_b^\pm(r),$$

where
$$\varphi_b^\pm(r) = \varphi_a^\pm(r)\int^r \frac{dx\,\Delta(x)}{[\varphi_a^\pm(x)]^2 x}$$

is the second solution of the equation $L^{(0)}\varphi = 0$ with the asymptotic representations $\varphi_b^+ \sim r^{-\frac{1}{2}} e^{-ikr}$ when $r \to \infty$ and $\varphi_b^- \sim r^{-m}$ when $r \to 0$. Taking account of the boundary conditions, we write $\rho^{(1)}$ as

$$\rho^{(1)\pm}(r) = \varphi_a^\pm(r)\int_{R^\pm}^r \frac{dx\,\Delta(x)}{x\varphi_a^{\pm 2}(x)} F^\pm(x) + a^\pm\varphi_a^\pm(r), \tag{3.16}$$

where
$$F(r) = \int_{R^\pm}^r Q^\pm(x)\varphi_a^\pm(x)\,dx. \tag{3.17}$$

Here we have denoted $R^- = 0$, $R^+ = \infty$, and $a^\pm(\tau)$ is so far an arbitrary 'constant'. With the aid of (3.16) and (2.18) we write the inner asymptotic expansion of the fundamental harmonic of the outer solution, by assuming $r - r_c = \mu Y$, as

$$\rho_1 = p_1^{(1)} + \mu p_1^{(2)} + \mu^2 p_1^{(3)} + \dots, \tag{3.18}$$

$$\left.\begin{aligned} p_1^{(1)} &= A, \\ p_1^{(2)} &= AqY(\hat{l} + B^\pm) + \tilde{a}^\pm - \frac{iq}{m\Omega_c'}\frac{\partial A}{\partial \tau}\hat{l}, \\ p_1^{(3)} &= [q_1 + qq_2(\hat{l} + B^\pm)]\,Y^2 A + q\left[\tilde{a}^\pm + \frac{iq}{m\Omega_c'}\frac{\partial A}{\partial \tau}B^\pm + \frac{i}{m\Omega_c'}\frac{\partial A}{\partial \tau}\left(\frac{f_c'}{f_c} - \frac{\Omega_c''}{\Omega_c'}\right)\right]\hat{l}Y \\ &\quad + \left[qB^\pm\tilde{a}^\pm - \left(\frac{\kappa^2}{r}\right)_c \mathrm{FP}\int_{R^\pm}^{r_c} Q\varphi_a^\pm\,dr - \frac{iq}{m\Omega_c'}\frac{\partial A}{\partial \tau}\left(\frac{f_c'}{f_c} - \frac{\Omega_c''}{\Omega_c'}\right)\right. \\ &\quad \left. - \frac{iq}{m\Omega_c'}\left(\frac{\kappa^2}{r}\right)_c\left(\frac{r}{\kappa^2}\right)_c'\frac{\partial A}{\partial \tau}\right] Y. \end{aligned}\right\} \tag{3.19}$$

Here we designate

$$\hat{l} = \ln|\mu Y|, \quad \tilde{a}^{\pm} = a^{\pm} - \left(\frac{\kappa^2}{r}\right)_{\rm c} \mathrm{FP} \int_{R^{\pm}}^{r_{\rm c}} \mathrm{d}r \frac{\Delta(r) F^{\pm}(r)}{r\varphi_a^{\pm 2}(r)}.$$

3.2.3. *The inner problem*

For the radial and azimuthal velocities and density we put

$$v_r = \epsilon V, \quad v_\varphi = \Omega r + \epsilon U, \quad \rho = 1 + \epsilon P. \tag{3.20}$$

We now write the original system of nonlinear equations in the vicinity of $r = r_{\rm c}$ using this notation, leaving only the terms which will be needed in the subsequent treatment:

$$\mu\left(\frac{\partial V}{\partial \tau} + \Omega'_{\rm c} Y \frac{\partial V}{\partial \varphi}\right) - 2(\Omega_{\rm c} + \Omega'_{\rm c}\mu Y)\,U + \frac{1}{M^2}\frac{1}{\mu}\frac{\partial P}{\partial Y} = 0, \tag{3.21}$$

$$\mu\left(\frac{\partial U}{\partial \tau} + \Omega'_{\rm c} Y \frac{\partial U}{\partial \varphi}\right) + \tfrac{1}{2}\Omega''_{\rm c}\mu^2 Y^2 \frac{\partial U}{\partial \varphi} + \left[\left(\frac{\kappa^2}{2\Omega}\right)_{\rm c} + \left(\frac{\kappa^2}{2\Omega}\right)'_{\rm c}\mu Y + \tfrac{1}{2}\mu^2 Y^2 \left(\frac{\kappa^2}{2\Omega}\right)''_{\rm c}\right] V$$

$$+ \frac{1}{M^2 r_{\rm c}}\left(1 - \mu\frac{Y}{r_{\rm c}} + \mu^2 \frac{Y^2}{r_{\rm c}^2}\right)\frac{\partial P}{\partial \varphi} = -\frac{\epsilon}{\mu} V \frac{\partial U}{\partial Y}, \tag{3.22}$$

$$\mu\left(\frac{\partial P}{\partial \tau} + \Omega'_{\rm c} Y \frac{\partial P}{\partial \varphi}\right) + \frac{1}{\mu}\frac{\partial V}{\partial Y} + \frac{1}{r_{\rm c}} V + \frac{1}{r_{\rm c}}\left(1 - \mu\frac{Y}{r_{\rm c}}\right)\frac{\partial U}{\partial \varphi} - \frac{\mu Y}{r_{\rm c}^2}\frac{\partial U}{\partial \varphi} = -\frac{\epsilon}{\mu}\frac{\partial}{\partial Y}(PV). \tag{3.23}$$

We expand the functions P, V, and U in harmonics similar to (3.8). It appears that the main contribution to the nonlinear evolution equation is made by the term on the right-hand side of (3.22), and this contribution gives an interaction of the fundamental ($l = 1$) harmonic V_1 with disturbances of the mean ($l = 0$) flow U_0. The contribution to the evolution equation resulting from the interaction of the fundamental ($l = -1$) harmonic with the second ($l = 2$) harmonic turns out to be of a smaller order of magnitude. Therefore, in our presentation we shall confine ourselves to the fundamental and zeroth harmonics. Analysis reveals that their expansions are representable as

$$\left.\begin{aligned} P_1 &= P_1^{(1)} + \mu P_1^{(2)} + \mu^2 P_1^{(3)} + \dots, \\ V_1 &= V_1^{(1)} + \mu V_1^{(2)} + \mu^2 V_1^{(3)} + \dots, \\ U_1 &= U_1^{(1)} + \mu U_1^{(2)} + \dots, \end{aligned}\right\} \tag{3.24}$$

$$P_0 = \frac{\epsilon}{\mu} P_0^{(1)} + \dots, \quad V_0 = \epsilon V_0^{(1)} + \dots, \quad U_0 = \frac{\epsilon}{\mu^2} U_0^{(1)} + \dots. \tag{3.25}$$

The desired evolution equation will be obtained by matching $P_1^{(3)}$ with the outer solution $p_1^{(3)}$ at $O(\mu^2)$ (or, equivalently, at $O(\epsilon^2/\mu^3)$.

Let us give briefly the results of consecutive iterations:

(i) *The fundamental*, $P_1^{(1)}$ *and* $V_1^{(1)}$

$$P_1^{(1)} = A(\tau), \tag{3.26}$$

$$V_1^{(1)} = -\frac{im}{M^2 r_{\rm c}}\left(\frac{2\Omega}{\kappa^2}\right)_{\rm c} A. \tag{3.27}$$

(ii) *The fundamental*, $P_1^{(2)}$, $V_1^{(2)}$ *and* $U_1^{(1)}$

$$\mathscr{L}_t P_1^{(2)''} = qA, \tag{3.28}$$

$$U_1^{(1)} = \frac{1}{2\Omega_c M^2} P_1^{(2)'}, \tag{3.29}$$

$$V_1^{(2)} = -\left(\frac{2\Omega}{\kappa^2}\right)_c \left\{\frac{im\Omega'_c}{2\Omega_c M^2}\mathscr{L}_t P_1^{(2)'} + \frac{im}{M^2 r_c}P_1^{(2)} - \frac{im}{M^2 r_c}AY\left[1 + r_c\left(\frac{2\Omega}{\kappa^2}\right)_c\left(\frac{\kappa^2}{2\Omega}\right)'_c\right]\right\}. \tag{3.30}$$

Here
$$\mathscr{L}_t = Y - \frac{i}{m\Omega'_c}\frac{\partial}{\partial \tau}, \tag{3.31}$$

and the primed P is the derivative in Y. By solving (3.28) with the condition that at $\tau \to -\infty$ all disturbances tend to zero, we find

$$P_1^{(2)''}(\tau, Y) = im\Omega'_c q \int_0^\infty e^{-im\Omega'_c Yt} A(\tau - t)\, dt. \tag{3.32}$$

One can show that the $P_1(\tau, Y)$ described by this formula is analytic in the upper half-plane of the complex variable Y (and even in the horizontal band of the lower half-plane $\mathrm{Im}(Y) > \gamma_L/m\Omega'_c$). We have adopted here the same philosophy of 'adiabatic inclusion' of the disturbance when $\tau \to -\infty$ as in Churilov & Shukhman (1988).† This permits us to avoid the difficulty associated with the need to specify the initial conditions not only for the fundamental but also for all the other harmonics if the Cauchy problem is formulated for $\tau = 0$.

Matching with the outer solution yields

$$Aq(B^+ - B^-) = \oint P_1^{(2)''}\, dY = Aq i\pi \,\mathrm{sign}\,(\Omega'_c),$$

whence we obtain the above stated relationship about the indentation rule from the above (when $\Omega'_c < 0$):

$$B^+ - B^- = -i\pi. \tag{3.33}$$

Also, from analyticity of $P_1^{(2)}$ in the upper half-plane it follows that

$$\tilde{a}^+ - \tilde{a}^- = -\frac{\pi q}{m\Omega'_c}\frac{\partial A}{\partial \tau}. \tag{3.34}$$

(iii) *The zeroth harmonics*, $U_0^{(1)}$, $P_0^{(1)}$, *and* $V_0^{(1)}$

We have

$$\frac{\partial U_0^{(1)}}{\partial \tau} = \frac{1}{q\Omega'_c M^4 r_c \kappa_c^2}\frac{\partial}{\partial \tau}|P_1^{(2)''}|^2, \tag{3.35}$$

$$V_0^{(1)} = 0, \tag{3.36}$$

$$P_0^{(1)'} = 2\Omega_c M^2 U_0^{(1)}. \tag{3.37}$$

From (3.32) and (3.35), we find:

$$U_0^{(1)}(\tau, Y) = \left(\frac{m}{M}\right)^2 \frac{\Omega'_c q}{r_c \kappa_c^2}\int_0^\infty dt_1 \int_0^\infty dt_2\, A(\tau - t_1)\bar{A}(\tau - t_2)\exp\left[-im\Omega'_c Y(t_1 - t_2)\right]. \tag{3.38}$$

Here the overbar denotes a complex conjugate.

† A conceptually similar hypothesis about the adiabatic inclusion of the disturbance when $x \to -\infty$ was adopted by G & L (1989) and Goldstein & Choi (1989) in their analysis of the spatial evolution of disturbances.

(iv) *The fundamental*, $P_1^{(3)}$, $V_1^{(3)}$, *and* $U_1^{(2)}$.

At this order we shall derive the desired evolution equation. We obtain

$$\mathscr{L}_t P_1^{(3)''} = \mathscr{R}_L + \mathscr{R}_N, \tag{3.39}$$

$$\mathscr{R}_L = qP_1^{(2)} + \mathscr{L}_t A\left[2q_1 + \tfrac{3}{2}q^2 - \tfrac{1}{2}q\left(\frac{\kappa^2}{r}\right)_c \left(\frac{r}{\kappa^2}\right)'_c\right]$$
$$+\frac{\mathrm{i}q}{m\Omega'_c}\frac{\partial A}{\partial \tau}\left[\frac{f'_c}{f_c} - \frac{\Omega''_c}{\Omega'_c} - \left(\frac{r}{\kappa^2}\right)'_c\left(\frac{\kappa}{r}\right)_c\right] + \frac{\Omega''_c}{2\Omega'_c}\frac{\partial^2}{\partial \tau^2}P_1^{(2)''} - \left(\frac{r}{\kappa^2}\right)'_c\left(\frac{\kappa^2}{r}\right)_c \mathscr{L}P_1^{(2)'}, \tag{3.40}$$

$$\mathscr{R}_N = -q\left(\frac{2\Omega}{\kappa^2}\right)_c \left(\frac{m}{M}\right)^4 \frac{\Omega_c'^2}{r_c^2} A(\tau)\int_0^\infty \mathrm{d}t_1 \int_0^\infty \mathrm{d}t_2$$
$$\times A(\tau - t_1)\bar{A}(\tau - t_2)(t_1 - t_2)^2 \exp\{-\mathrm{i}m\Omega'_c Y(t_1 - t_2)\}. \tag{3.41}$$

Here on the right-hand side (3.39) two contributions are distinguished: linear $\mathscr{R}_L$ and nonlinear $\mathscr{R}_N$, which are responsible for the linear and nonlinear parts of the evolution equation, respectively. It should be recalled that here the entire nonlinear contribution $\mathscr{R}_N$ is due to the nonlinear term $V_1\,\partial U_0/\partial Y$ in (3.22).

Next, it is necessary to perform the matching to the outer solution. From (3.39) we find the asymptotic expansion $P_1^{(3)'}$ as $Y \to \pm\infty$:

$$P_1^{(3)'} \sim (2q_1 + qq_2)AY + 2qq_2 AY(\ln\mu Y + B^+)$$
$$+\left[\frac{\mathrm{i}q}{m\Omega'_c}\frac{\partial A}{\partial \tau}\left(\frac{f'_c}{f_c} - \frac{\Omega''_c}{\Omega'_c} - q\right) + q\tilde{a}^+ + \frac{\mathrm{i}q}{m\Omega'_c}\frac{\partial A}{\partial \tau}(1 + B^+)\right]\ln\mu Y + C^\pm, \tag{3.42}$$

where $\ln(\mu Y)$ is understood in the sense of indentation from above and the jump $C^+ - C^-$ is associated with non-analyticity of $\mathscr{R}_N$ in the upper half-plane (for $\mathscr{R}_N$ analytic in the upper half-plane, there would be no jump C and this is why the contribution from the second harmonic disappears at the order under consideration). We have

$$C^+ - C^- = ⨍ P_{1N}^{(3)''}\,\mathrm{d}Y, \tag{3.43}$$

where
$$⨍_{-\infty}^{\infty}(\ldots)\,\mathrm{d}Y = \lim_{L\to\infty}\int_{L}^{L}(\ldots)\,\mathrm{d}Y,$$

and
$$\mathscr{L}_t P_{1N}^{(3)'} = \mathscr{R}_N. \tag{3.44}$$

By solving this equation in a manner similar to (3.28), we obtain

$$⨍_{-\infty}^{\infty} P''_N(\tau, Y)\,\mathrm{d}Y = \mathrm{i}m\Omega'_c \int_0^\infty \mathrm{d}t \int_{-\infty}^{\infty} \mathscr{R}_N(Y, \tau - t)\,\mathrm{e}^{-\mathrm{i}m\Omega'_c Yt}\,\mathrm{d}Y. \tag{3.45}$$

We substitute (3.41) into (3.45) and integrate over Y:

$$C^+ - C^- = \frac{2\mathrm{i}\pi}{r_c^2}\left(\frac{2\Omega}{\kappa^2}\right)_c \left(\frac{m}{M}\right)^4 \Omega_c'^2 q \int_0^\infty t^3\,\mathrm{d}t \int_0^1 \sigma^2\,\mathrm{d}\sigma A(\tau - t)A(\tau - \sigma t)\bar{A}(\tau - (1 + \sigma)t). \tag{3.46}$$

By matching $P_1^{(3)'}$ with $p_1^{(3)'}$ from (3.19) and taking account of (3.33) and (3.34), we find

$$C^+ - C^- = \frac{\mathrm{i}q}{m\Omega'_c}\frac{\partial A}{\partial \tau}\left\{q(2\mathrm{i}\pi B^- + \pi^2) + \mathrm{i}\pi\left(\frac{f'_c}{f_c} - \frac{\Omega''_c}{\Omega'_c}\right)\right\} + \left(\frac{\kappa^2}{r}\right)_c \mathrm{FP}\int_0^\infty Q(r)\varphi_a(r)\,\mathrm{d}r. \tag{3.47}$$

The right-hand side of (3.47) is none other than the integral $\int_C Q\varphi_a\,\mathrm{d}r(\kappa^2/r)_c$ with indentation from above (see (2.21)). Substitute (3.46) into (3.47); then using the notation of (2.10) and returning to the 'physical' time variables $t = \tau/\mu$ and to the pressure amplitude $\tilde{A} = \epsilon M^{-2}A$ and again denoting $\tilde{A}$ by A, we write the evolution equation as

$$\frac{\mathrm{i}}{m}I_t\frac{\partial A}{\partial t}+\delta M I_M A = 2\pi \mathrm{i} a\int_0^\infty \tau^3\,\mathrm{d}\tau\int_0^1 \sigma^2\,\mathrm{d}\sigma A(t-\tau)A(t-\sigma\tau)\bar{A}(t-(1+\sigma)\tau), \tag{3.48}$$

where
$$a=\left(\frac{2\Omega_{\mathrm{c}}m}{\kappa^2}\right)^4_{\mathrm{c}}\left(\frac{\kappa^2}{2\Omega}\right)'_{\mathrm{c}}\frac{\Omega'_{\mathrm{c}}}{r_{\mathrm{c}}^2}>0. \tag{3.49}$$

We defer the analysis of this equation to §4, first deriving the evolution equation in the regime with a viscous CL.

3.3. *The evolution equation in the regime with a viscous CL*

3.3.1. *Scaling*

According to §3.1, we put

$$\frac{\partial}{\partial t}=\mu\frac{\partial}{\partial\tau}-\Omega_{\mathrm{p}}\frac{\partial}{\partial\varphi},\quad \delta M=\mu M_1,\quad \nu=\tilde{\eta}\lambda^3, \tag{3.50}$$

where $\tilde{\eta}=O(1)$, and $\lambda \ll 1$ is the scale of a viscous CL, and in this case $\mu=\epsilon^2/\lambda^4$.

3.3.2. *The outer problem*

There are no substantial differences here from the case of an unsteady CL; however, another ordering arises:

$$\rho_1 = p^{(1)}+\lambda p_1^{(2)}+\mu p_1^{(3)}+\lambda^2 p_1^{(4)}+\lambda\mu p_1^{(5)}+\dots, \tag{3.51}$$

where now

$$\left.\begin{aligned}
&p_1^{(1)}=A,\quad p_1^{(2)}=AqY(\hat{l}+B^\pm),\\
&p_1^{(3)}=\tilde{a}^\pm-\frac{\mathrm{i}q}{m\Omega'_{\mathrm{c}}}\frac{\partial A}{\partial\tau}\hat{l},\quad p_1^{(4)}=[q_1+qq_2(\hat{l}+B^\pm)]\,Y^2A,\\
&p_1^{(5)}=q\left\{\tilde{a}^\pm+\frac{\mathrm{i}q}{m\Omega'_{\mathrm{c}}}B^\pm\frac{\partial A}{\partial\tau}+\frac{\mathrm{i}}{m\Omega'_{\mathrm{c}}}\frac{\partial A}{\partial\tau}\left(\frac{f'_{\mathrm{c}}}{f_{\mathrm{c}}}-\frac{\Omega''_{\mathrm{c}}}{\Omega'_{\mathrm{c}}}\right)\right]lY\\
&\qquad+\left[qB^\pm\tilde{a}^\pm-\left(\frac{\kappa^2}{r}\right)_{\mathrm{c}}\mathrm{FP}\int_{R^\pm}^{r_{\mathrm{c}}}Q\varphi_a\,\mathrm{d}r-\frac{\mathrm{i}q}{m\Omega'_{\mathrm{c}}}\frac{\partial A}{\partial\tau}\left(\frac{f'_{\mathrm{c}}}{f_{\mathrm{c}}}-\frac{\Omega''_{\mathrm{c}}}{\Omega'_{\mathrm{c}}}\right)\right.\\
&\qquad\left.-\frac{\mathrm{i}q}{m\Omega'_{\mathrm{c}}}\left(\frac{\kappa^2}{r}\right)_{\mathrm{c}}\left(\frac{r}{\kappa^2}\right)'_{\mathrm{c}}\frac{\partial A}{\partial\tau}\right]Y.
\end{aligned}\right\} \tag{3.52}$$

Here we have put $r-r_{\mathrm{c}}=\lambda Y, \hat{l}=\ln|\lambda Y|$.

3.3.3. *The inner problem*

In the notation of (3.20) we write the original system of equations in the vicinity of $r=r_{\mathrm{c}}$, again retaining only those terms which are needed in what follows:

$$\lambda\left(\Omega'_{\mathrm{c}}Y\frac{\partial}{\partial\varphi}-\tilde{\eta}\frac{\partial^2}{\partial Y^2}\right)V-2(\Omega_{\mathrm{c}}+\Omega'_{\mathrm{c}}\lambda Y)\,U+\frac{1}{M^2}\frac{1}{\lambda}\frac{\partial P}{\partial Y}=-\mu\frac{\partial V}{\partial\tau}, \tag{3.53}$$

$$\begin{aligned}
&\lambda\left(\Omega'_{\mathrm{c}}Y\frac{\partial}{\partial\varphi}-\tilde{\eta}\frac{\partial^2}{\partial Y^2}\right)U+\tfrac{1}{2}\Omega''_2\lambda^2Y^2\frac{\partial U}{\partial\varphi}+\left[\left(\frac{\kappa^2}{2\Omega}\right)_{\mathrm{c}}+\left(\frac{\kappa^2}{2\Omega}\right)'_{\mathrm{c}}\lambda Y\right.\\
&\quad\left.+\frac{1}{2}\left(\frac{\kappa^2}{2\Omega}\right)''_{\mathrm{c}}\lambda^2Y^2\right]V+\frac{1}{M^2r_{\mathrm{c}}}\left(1-\frac{\lambda}{r_{\mathrm{c}}}Y+\frac{\lambda^2}{r_{\mathrm{c}}^2}Y^2\right)\frac{\partial P}{\partial\varphi}=-\mu\frac{\partial U}{\partial\tau}+\tilde{\eta}\frac{\lambda^2}{r_{\mathrm{c}}}\frac{\partial U}{\partial Y}-\frac{\epsilon}{\lambda}V\frac{\partial U}{\partial Y},
\end{aligned} \tag{3.54}$$

$$\frac{1}{\lambda}\frac{\partial V}{\partial Y}+\frac{1}{r_{\mathrm{c}}}\left(1-\frac{\lambda}{r_{\mathrm{c}}}Y\right)V+\frac{1}{r_{\mathrm{c}}}\left(1-\frac{\lambda}{r_{\mathrm{c}}}Y\right)\frac{\partial U}{\partial\varphi}+\Omega'_{\mathrm{c}}\lambda Y\frac{\partial P}{\partial\varphi}=-\mu\frac{\partial P}{\partial\tau}-\frac{\epsilon}{\lambda}\frac{\partial}{\partial Y}(PV). \tag{3.55}$$

Here the main nonlinear contribution is made by the last term on the right-hand side of (3.54). For the case considered here it is necessary to adopt a new ordering of the harmonics:

$$P_1 = P_1^{(1)} + \lambda P_1^{(2)} + \mu P_1^{(3)} + \lambda^2 P_1^{(4)} + \lambda\mu P_1^{(5)} + \dots,$$
$$V_1 = V_1^{(1)} + \lambda V_1^{(2)} + \mu V_1^{(3)} + \lambda^2 V_1^{(4)} + \lambda\mu V_1^{(5)} + \dots,$$
$$U_1 = U_1^{(1)} + \lambda^{-1}\mu U_1^{(2)} + \lambda U_1^{(3)} + \mu U_1^{(4)} + \dots,$$
$$P_0 = P_0^{(1)}\frac{\epsilon}{\lambda} + \dots, \quad V_0 = V_0^{(1)}\epsilon + \dots, \quad U_0 = U_0^{(1)}\epsilon^2/\lambda^2 + \dots.$$

The evolution equation is obtained by matching $P_1^{(5)}$ to $p_1^{(5)}$ at order $O(\lambda\mu) = O(\epsilon^2/\lambda^2)$ at the fifth step of the iteration process for P_1. The second harmonic, as before, is not needed.

Omitting the subsequent derivation, we shall write only the expression for the zeroth harmonic of azimuthal velocity U_0 which replaces (3.38) for the case of a viscous CL:

$$[U_0^{(1)}(\tau, Y)]'' = -\frac{\mathrm{i}m}{M^4}\frac{sq}{r_c\kappa_c^2}\tilde{\eta}^{-\frac{4}{3}}|A|^2\left\{\phi\left(\frac{sY}{\tilde{\eta}^{\frac{1}{3}}}\right) - \text{c.c.}\right\}, \tag{3.56}$$

where we designated $s = |m\Omega_c'|^{\frac{1}{3}}$, and $\phi(x)$ is a function known from previous papers (see e.g. Churilov & Shukhman 1987*a*) satisfying the equation $\phi''(x) - \mathrm{i}x\phi(x) = -\mathrm{i}$ with the asymptotic representation in the region $\mathscr{M}$ (i.e. when $|x| \to \infty$) $-\frac{7}{6}\pi \leqslant \arg(x) \leqslant \frac{1}{6}\pi$.

Finally, we write the nonlinear equation in the regime with a viscous CL (in 'physical' variables) as

$$\frac{\mathrm{i}}{m}I_t\frac{\mathrm{d}A}{\mathrm{d}t} + \delta M I_M A = \mathrm{i}a\pi(\tfrac{2}{3})^{\frac{2}{3}}\Gamma(\tfrac{1}{3})(s^2\nu^{\frac{1}{3}})^{-4}A|A|^2. \tag{3.57}$$

The appearance of the numerical coefficient in (3.57) is associated with the integral

$$\int_{-\infty}^{\infty}|\phi(x)|^2\,\mathrm{d}x = \pi(\tfrac{2}{3})^{\frac{2}{3}}\Gamma(\tfrac{1}{3}).$$

The expression for a is given by (3.49).

4. The analysis of solutions of the evolution equations

Let us determine the character of solutions described by equations (3.57) for a viscous CL and (3.48) for an unsteady CL. Let us begin by considering the simpler, viscous case.

4.1. *The evolution of disturbances in the regime with a viscous CL*

Through the replacement $A = B\exp(-\mathrm{i}m\Delta\Omega_p t)(|I_t|\alpha m)^{\frac{1}{2}}$, where

$$\alpha = a\pi(\tfrac{2}{3})^{\frac{2}{3}}\Gamma(\tfrac{1}{3})(s^2\nu^{\frac{1}{3}})^{-4}, \quad \Delta\Omega_p = -\mathrm{Re}\,(I_M/I_t)\,\delta M,$$

equation (3.57) is reduced to the form:

$$\frac{\mathrm{d}B}{\mathrm{d}t} - \gamma_L B = \mathrm{e}^{-\mathrm{i}\psi}B|B|^2, \tag{4.1}$$

where $\exp(\mathrm{i}\psi) = I_t/|I_t|$. The solutions of (4.1) (the so-called Stuart–Watson–Landau equation) are well known; therefore, let us recall only briefly the results. Depending on the sign of $\mathrm{Re}\,(\mathrm{e}^{\mathrm{i}\psi})$, three cases are possible.

(i) $\cos\psi > 0$.

Here a solution of the explosive type arises with the asymptotic behaviour

$$B = B_*(t_0-t)^{-\frac{1}{2}+\frac{1}{2}\mathrm{i}\tan\psi} \quad \text{where} \quad |B_*|^2 = \frac{1}{2\cos\psi}. \tag{4.2}$$

(ii) $\cos\psi < 0$.

The asymptotic solution has a finite, time-independent, amplitude and phase increasing linearly in time (the extreme point of vector B on the complex plane B uniformly rotates in a circle)

$$B = B_* \exp(\mathrm{i}\gamma_\mathrm{L} t \tan\psi), \quad |B_*|^2 = \gamma_\mathrm{L}|\cos\psi|^{-1}. \tag{4.3}$$

(iii) $\psi = \pm\frac{1}{2}\pi$.

Here the nonlinear term is a purely imaginary one, and we have simply a nonlinear correction to the frequency:

$$B = B_0 \exp\left\{\mp\frac{\mathrm{i}B_0^2}{2\gamma_\mathrm{L}}\mathrm{e}^{2\gamma_\mathrm{L}t}+\gamma_\mathrm{L}t\right\} \equiv B_0\exp(\gamma_\mathrm{L}t)\exp\left(\mp\mathrm{i}\int|B|^2\,\mathrm{d}t\right). \tag{4.4}$$

The value of $\psi = \arg(I_t/|I_t|)$ is wholly determined by the solution of the linear problem and, more exactly, by the argument of the complex integral I_t. A numerical calculation of I_t shows that ψ lies within the interval from 0 to $\frac{1}{2}\pi$, i.e. case (i) is realized, which corresponds to the explosive instability $|B| \sim (t_0-t)^{-\frac{1}{2}}$.

4.2. *The evolution of disturbances in the regime with an unsteady CL*

We again take $A = B\exp(-\mathrm{i}m\Delta\Omega_\mathrm{p}t)(|I_t|/2\pi ma)^{\frac{1}{2}}$ and obtain

$$\frac{\mathrm{d}B}{\mathrm{d}t}-\gamma_\mathrm{L}B = \mathrm{e}^{-\mathrm{i}\psi}\int_0^\infty \tau^3\,\mathrm{d}\tau\int_0^1\sigma^2\,\mathrm{d}\sigma\,B(t-\tau)\,B(t-\delta\tau)\,\bar{B}(t-(1+\sigma)\,\tau). \tag{4.5}$$

By analogy with the viscous CL it seems that if $\mathrm{Re}(\mathrm{e}^{\mathrm{i}\psi}) > 1$, there will be an explosive regime. It appears that this is indeed the case and, moreover, the explosive regime also occurs when $\mathrm{Re}(\mathrm{e}^{\mathrm{i}\psi}) < 0$. Even with $\mathrm{Re}(\mathrm{e}^{\mathrm{i}\psi}) = -1$ ($\psi = \pi$) when the nonlinear term has a 'stabilizing' sign, an unlimited growth of amplitude occurs. Such unusual behaviour of the solution is associated with the integro-differential character of (3.48) which distinguishes it substantially from the time-local equation (3.57). Here the amplitude 'remembers' all its past history.

It is easy to determine the asymptotic law of the explosive solution. By dropping at the explosive stage the term $\gamma_\mathrm{L}B$ in (4.5), we obtain

$$B = B_*(t_0-t)^{-\frac{5}{2}+\mathrm{i}\beta}. \tag{4.6}$$

For determining B_* and β, we have

$$\tfrac{5}{2}-\mathrm{i}\beta = |B_*|^2\,\mathrm{e}^{-\mathrm{i}\psi}I(\beta), \tag{4.7}$$

where
$$I(\beta) = \int_0^1 \mathrm{d}\sigma\,\sigma^3(1-\sigma)^{\frac{5}{2}-\mathrm{i}\beta}\int_0^1[1-\sigma(1-x)]^{-\frac{5}{2}+\mathrm{i}\beta}(1+\sigma x)^{-\frac{5}{2}-\mathrm{i}\beta}x^2\,\mathrm{d}x.$$

The parameter $\beta = \beta(\psi)$ will be determined from (4.7):

$$\mathrm{Im}\,\{I(\beta)\,\mathrm{e}^{-\mathrm{i}\psi}(\tfrac{5}{2}+\mathrm{i}\beta)\} = 0. \tag{4.8}$$

We must choose the roots β of equation (4.8), for which

$$j \equiv \mathrm{Re}\,\{I(\beta)\,\mathrm{e}^{-\mathrm{i}\psi}(\tfrac{5}{2}+\mathrm{i}\beta)\} > 0. \tag{4.9}$$

Then
$$|B_*|^2 = j^{-1}[(\tfrac{5}{2})^2+\beta^2].$$

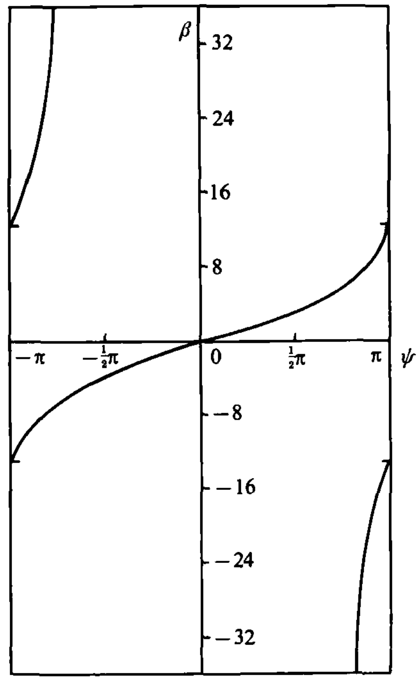

FIGURE 4. The dependence of β on ψ for the asymptotic solution in the unsteady CL regime: $B \sim (t_0 - t)^{-\frac{5}{2}+\mathrm{i}\beta(\psi)}$.

Equation (4.8) with the condition (4.9) was solved numerically. The result is shown in figure 4 and equivalent results appear in figure 1 of G & L. The function $\beta(\psi)$ is single-valued when $|\psi| < \frac{1}{2}\pi$ and is double-valued when $\frac{1}{2}\pi < \psi < \pi$ and $-\pi < \psi < -\frac{1}{2}\pi$. A numerical solution of the complete equation (4.5) for different ψ shows that there is a branch of the plot through the origin.

Some words about the case $\psi = \pm\pi$ are in order. It would seem that here β must be zero because the initial equation for $\psi = \pm\pi$ is purely real. Indeed, when $\psi = \pm\pi$ the solution is not described by formula (4.6); however, when ψ slightly differs from $\pm\pi$, the solution behaves according to (4.6).

In order to confirm the asymptotic solution (4.6) and to determine the moment of 'explosion' t_0, it is necessary to solve the complete equation (4.5). It can be reduced to a universal form, i.e. the form which contains a unique parameter ψ.

Let us make a substitution in (4.5): $B(t) = b(t)\exp(\gamma_L t)$. When $t \to -\infty$, $B(t)$ has the asymptotic behaviour $B(t) = b_0 \exp(\gamma_L t)$; therefore, the asymptotic behaviour for $b(t)$ is thus: $b(t) = b_0$ when $t \to -\infty$. Let us now introduce the 'logarithmic' time

$$T = |B_0|^2 \exp(2\gamma_L t)(2\gamma_L)^{-4}$$

and a new amplitude $$C = b_0/b.$$

By considering C to be a function of T, we obtain

$$\frac{\mathrm{d}C}{\mathrm{d}T} = \mathrm{e}^{-\mathrm{i}\psi}\int_0^\infty x^3 \mathrm{e}^{-x}\,\mathrm{d}x \int_0^1 \frac{\sigma^2\,\mathrm{d}\sigma}{(1+\sigma)^4} C(T\mathrm{e}^{-(x/(1+\sigma))})C(T\mathrm{e}^{-(x\sigma/(1+\sigma))})\,\bar{C}(T\mathrm{e}^{-x}), \quad (4.10)$$

with the initial condition $C(T=0) = 1$.

Equation (4.10) was solved numerically for five values of ψ: 0, $\frac{1}{4}\pi$, $\frac{1}{2}\pi$, 0.95π, and π. When $\psi = 0$ and π, the amplitude C remains real throughout the entire evolution and can be represented on the plots as a function of T. For $\psi = 0$, the amplitude

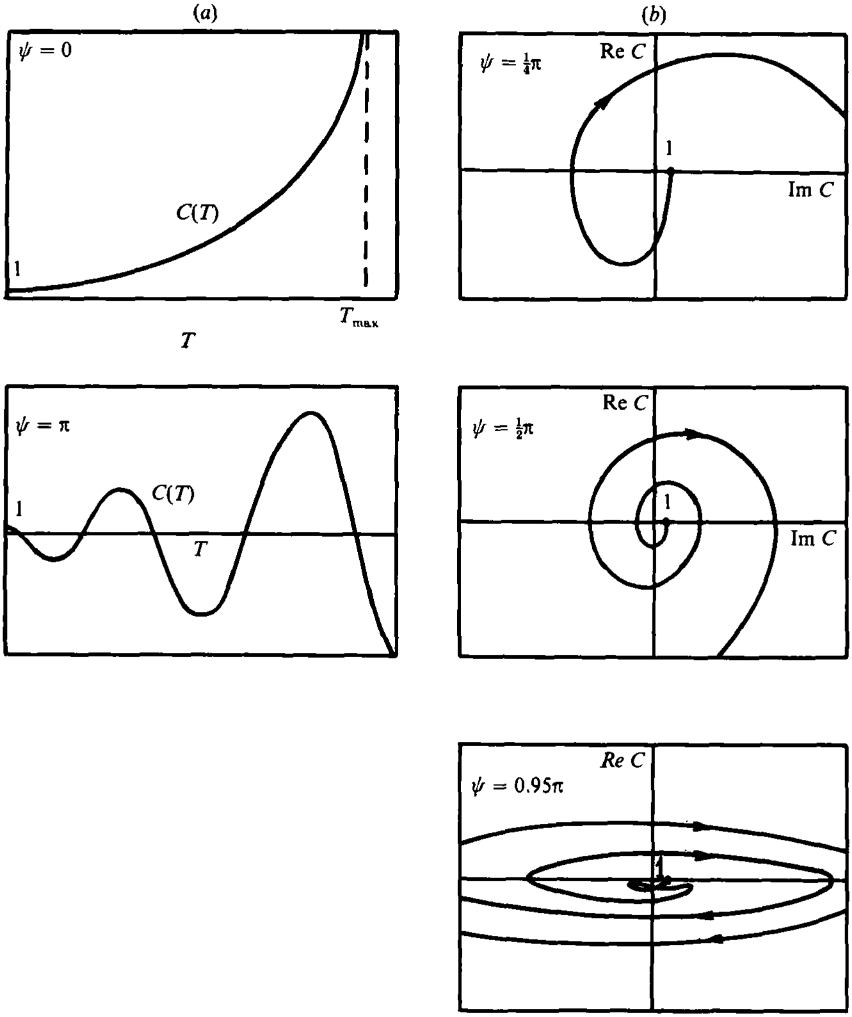

FIGURE 5. The time dependence of the $C(T)$ amplitude in the unsteady CL regime at different values of the parameter ψ. (a) $C(T)$ for $\psi = 0$ and π. Here $C(T)$ is real. (b) Evolution paths of the C amplitude on a complex plane for $\psi = \frac{1}{4}\pi$, $\frac{1}{2}\pi$, and 0.95π. (The results given here have been obtained through a numerical solution, but in the figure they are reproduced only schematically.)

increases monotonically and goes to infinity when $T \approx 31$. When $\psi = \pi$, $C(T)$ performs oscillations with increasing swing, whose amplitude grows without bounds when $T \to \infty$. For $\psi = \frac{1}{4}\pi$, $\frac{1}{2}\pi$, and 0.95π, C is complex and it is more convenient to represent it in parametric form on a complex plane C: $(\operatorname{Im} C(T), \operatorname{Re} C(T))$. Curves traced by the extreme point of vector C represent untwisting spirals, which start at point $C = 1$. As one would expect, the smaller ψ is, the earlier the solution reaches the explosive regime (singularity for $\psi = 0$, $\frac{1}{4}\pi$, and $\frac{1}{2}\pi$ is reached when $T \approx 31$, 38, and 71, respectively). When $\psi = 0.95\pi$, the spiral is very much flattened towards the imaginary axis (i.e. in general, the imaginary part is small compared with the real one). This agrees with the suggestion of a limiting transition to the case $\psi = \pi$. The singularity here is attained very late ($T > 1000$).

The qualitative behaviour of the curves is shown in figure 5. Remember that in our problem the values of ψ calculated from linear theory lie within the range $0 < \psi < \frac{1}{2}\pi$.

As has already been pointed out in the Introduction, an analysis of the evolution equation was made by this author and by Goldstein & Leib (1989) independently of each other. The difference between the two analyses lies in the fact that we have been able to represent the results in a more universal form. This is achieved thus: we reduced a two-parametric equation (4.5) to a one-parametric equation (4.10) by introducing a 'logarithmic' time. In this case the character of the evolution is uniquely defined by the quantity ψ only.

5. Conclusions

Let us now construct an overall picture of the evolution of an originally small disturbance. According to the previous discussion, there are two scenarios for the evolution, which are determined by the initial value of γ_L.

(1) $\gamma_L < \nu^{\frac{1}{3}}$

In this case there are three stages in the development of a disturbance.

(i) $|A| \leqslant (\gamma_L \nu^{\frac{4}{3}})^{\frac{1}{2}}$. The evolution proceeds according to linear theory:

$$|A| \sim \exp(\gamma_L t).$$

(ii) $(\gamma_L \nu^{\frac{4}{3}})^{\frac{1}{2}} < |A| < \nu^{\frac{5}{6}}$. In this stage, an explosive growth of amplitude occurs according to the law

$$|A| \sim (t_0 - t)^{-\frac{1}{2}}.$$

(iii) $|A| > \nu^{\frac{5}{6}}$. The explosive growth continues, but according to a faster law

$$|A| \sim (t_1 - t)^{-\frac{5}{2}}.$$

Stages (i) and (ii) occur in the regime with a viscous CL, and stage (iii) in the regime with an unsteady CL.

(2) $\gamma_L > \nu^{\frac{1}{3}}$

There are two evolutionary stages, both occurring in the regime with an unsteady CL.

(i) $|A| \leqslant \gamma_L^{\frac{5}{2}}$. Linear theory holds here:

$$|A| \sim \exp(\gamma_L t).$$

(ii) $|A| > \gamma_L^{\frac{5}{2}}$. An explosive growth of amplitude occurs according to the law

$$|A| \sim (t_1 - t)^{-\frac{5}{2}}.$$

We have illustrated this evolution on the amplitude–supercriticality diagram (figure 3) with the aid of arrows of different forms.

Thus, in this paper it has been shown that even with a small initial supercriticality the spiral wave amplitude reaches a small value, and stabilization of the instability can occur only at the level $\delta\rho/\rho \sim O(1)$.

Note that instead of two nonlinear equations corresponding to two CL regimes, viscous and unsteady, one can derive one general equation that can describe both regimes at once:

$$\frac{\mathrm{i}}{m} I_t \frac{\mathrm{d}A}{\mathrm{d}t} + \delta M I_M A = \mathrm{i} a N, \tag{5.1}$$

where

$$N = 2\pi \int_0^\infty \tau^3 \,\mathrm{d}\tau \int_0^1 \sigma^2 \,\mathrm{d}\sigma A(t-\sigma\tau) A(t-\tau) \bar{A}(t-(1+\sigma)\tau) \exp\{-\tfrac{1}{3}\nu s^3 \sigma^2 (3-\sigma)\tau^3\},$$
$$s = |m\Omega_c'|^{\frac{1}{3}}. \tag{5.2}$$

It is easy to show that in the two limiting cases corresponding to a viscous CL (when the integral has no contribution from the 'past') and an unsteady CL (when one can put $\nu = 0$), we obtain, respectively, (3.57) and (3.48). However, this general equation (5.1) defies a simple analysis such as we have been able to make. Besides, its derivation is substantially more cumbersome. Therefore, we have given preference to the form of presentation given in §3.

Note also that throughout this paper we have varied M at fixed D. The opposite is also possible. In this case, only the explicit expressions γ_L and $\Delta\Omega_p$ are changed (in (2.16) and (2.17) $I_M \delta M$ should be replaced by $I_D \delta D$).

In Hickernell's (1984) paper also an attempt is made to analyse the evolution of disturbances in a situation with a singular point. For that purpose, a model of a zonal shear flow on the β-plane was chosen. Similar to our condition $(\kappa^2/2\Omega)'_c \neq 0$ it was assumed that the derivative of absolute viscosity at the critical level does not go to zero: $(U''-\beta)_c \neq 0$ (the case with $(u''-\beta)_c = 0$ is considered in Churilov & Shukhman 1986, 1987*b* and Churilov 1989). The equation obtained in Hickernell (1984) is similar to our equation (5.1) and to equation (5.7) in G & L. The difference between Hickernell's equation and ours is associated only with the fact that he specified the initial conditions for $t = 0$, instead of the formulation with 'adiabatic inclusion' of disturbances when $t \rightarrow -\infty$ adopted by us. This complicated the character of the nonlinear term in his equation, and it seems likely that it is for that reason that he was unable to obtain any specific results on the character of the evolution of disturbances.

A comparison of our results with those reported by Hickernell demonstrates the remarkable property of the universality of nonlinear evolution equations in those problems where the main nonlinearity is attributable to processes occurring in a critical layer. It appears that the form of the nonlinear terms of these equations depends only on the character of the resonant point and does not depend on the structure of the flow as a whole. The entire difference in amplitude equations for such (at first glance, dissimilar) physical models as a compressible circular shear layer and a zonal incompressible flow on the β-plane is contained in the linear part of the problem and is reduced to specifying the phase $\psi = \arg(I_t/|I_t|)$. However, a still more surprising factor emerges when comparing Hickernell's and my models, on the one hand, and G & L's model, on the other. It appears that the evolution equation obtained involves an even greater universality than might be anticipated, if judging only from the analysis of peculiarities of eigenfunctions of the neutral mode. While for our case and that of Hickernell's these peculiarities are the same, in the G & L case the eigenfunctions have a totally different type of singularity at the critical level and, accordingly, a different structure of the critical layer. Nevertheless, the final evolution equation has the same form.

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Appendix. Calculating the growth rate and frequency of the $m = 1$ mode for small Mach numbers and for arbitrary values of $D < 1$

We proceed from the equation for disturbed pressure (2.3):

$$\frac{1}{D^2}(1-z^2)\frac{\partial}{\partial z}\frac{1-z^2}{\Delta(\Omega_p,z)}\frac{\partial p}{\partial z}-\left\{\frac{1}{\Delta}+\frac{1}{D}\frac{1-z^2}{\Delta^2}\left[-\left(\Omega_p+\frac{1-z}{2}\right)\right.\right.$$
$$\left.\left.+\frac{(1-z^2)(z/D-1)}{\Omega_p-\frac{1}{2}(1-z)}\right\}p=-M^2\left(\frac{1-z}{1+z}\right)^{-D}p. \quad \text{(A 1)}$$

Here

$$r=e^y=\left[\frac{(1-z)}{(1+z)}\right]^{-\frac{1}{2}D},\quad z=\tanh\left(\frac{y}{D}\right),\quad \Delta(\Omega_p,z)=\left(\Omega_p-\frac{1-z}{2}\right)^2-(1-z)^2\left(1-\frac{1+z}{2D}\right).$$

Note that when $M = 0$, (A 1) has the solution for an arbitrary Ω_p:

$$p=\left(\Omega_p^2-\frac{(1-z)^2}{4}\right)\left(\frac{1-z}{1+z}\right)^{-\frac{1}{2}D} \quad \text{(A 2)}$$

(or, equivalently, but in other variables $p=(\Omega_p^2-\Omega^2(r)\,r)$ and the solution satisfying the boundary conditions, requires that $\Omega_p = 0$.

We now put

$$\Omega_p=\epsilon\omega,\quad M=\epsilon^{1-\frac{1}{2}D}\mu,\quad \epsilon\ll 1. \quad \text{(A 3)}$$

Such a scaling, as will be shown below, must be chosen for correct matching. The point here is that the standard procedure of perturbation theory does not apply and it is necessary to apply the method of matched asymptotic expansions. Let us establish what asymptotic representations for p occur in different regions. When $r\to\infty$, or when $z\to 1$, (A 1) has the asymptotic solution $p\sim K_1(M\Gamma r)$, where K is MacDonald's function, and $\Gamma=-i\Omega_p$. When $r\ll 1/M\Gamma\sim(1/\mu\omega)\,\epsilon^{-2+\frac{1}{2}D}$, or when $(1-z)\gg\epsilon^{4/D-1}$ the asymptotic p is

$$p\sim r^{-1}\sim(1-z)^{\frac{1}{2}D}. \quad \text{(A 4)}$$

Another (characteristic on the axis $(1-z)$) point is defined by the condition $\Omega_p\sim\Omega$, or $(1-z)\sim\epsilon$. Further, when $r\to 0$, or when $z\to -1$

$$p\sim r+O(r^3)\sim(1+z)^{\frac{1}{2}D}+O(M^2r^3). \quad \text{(A 5)}$$

Thus, the axis $(1-z)$ is now divided into three regions: I, $0\leqslant(1-z)<\epsilon^{4/D-1}$; II, $\epsilon^{4/D-1}<(1-z)\lesssim\epsilon$, and III, $\epsilon\lesssim(1-z)\leqslant 2$. Region I will no longer play any role in the calculations to follow. The task is to find a solution in the region $1-z\sim O(\epsilon)$ that will match, on the left, with (A 4), and on the right with a solution of (A 1) in region III that has a correct asymptotic representation when $r\to 0$ $(1-z\to 2)$, i.e. (A 5).

We start by determining the solution in III when $1-z\gg\epsilon$. Here $(1-z)\gg\Omega_p$ and one can proceed according to usual perturbation theory. We get

$$p=p_0+\epsilon p_1+\epsilon^2 p_2+\epsilon^{2-D}\mu^2 p_3+\dots, \quad \text{(A 6)}$$

where

$$\left.\begin{aligned} p_0&=-\tfrac{1}{4}(1-z^2)\left(\frac{1-z}{1+z}\right)^{-\frac{1}{2}D}A,\\ p_1&=0,\quad p_2=\omega^2 A\left(\frac{1-z}{1+z}\right)^{-\frac{1}{2}D}, \end{aligned}\right\} \quad \text{(A 7)}$$

$$p_3=-p_0D^2\int_{-1}^{z}\left(\frac{1+z_1}{2D}-\frac{3}{4}\right)(1-z_1)^{D-3}(1+z_1)^{-1-D}\,dz_1\int_{-1}^{z_1}(1-z_2)^{3-2D}(1+z_2)^{2D-1}\,dz_2. \quad \text{(A 8)}$$

The limits of integration in (A 8) are chosen such that, when $z \to -1$, the impurity of the solution $(1+z)^{-\frac{1}{2}D} \sim 1/r$ which arises with a different choice, should be excluded. We have

$$p_3 \sim (1+z)^{3D/2} \sim r^3 \quad \text{when} \quad z \to -1,$$
$$p_3 = k_1(1-z)^{\frac{1}{2}D}A \quad \text{when} \quad z \to 1,$$

where
$$k_1 = -2^{-3-\frac{1}{2}D}\frac{D^2}{D-2}\int_{-1}^{1}(1-x)^{3-2D}(1+x)^{2D-1}\,dx. \tag{A 9}$$

Thus, we write now the asymptotic expansion (A 6), when $(1-z) \ll 1$, in explicit form

$$p = A\{-\tfrac{1}{4}2^{\frac{1}{2}D}(1-z)^{2-\frac{1}{2}D}+2^{\frac{1}{2}D}\epsilon^2\omega^2(1-z)^{-\frac{1}{2}D}+\epsilon^{2-D}\mu^2k_1(1-z)^{\frac{1}{2}D}\}. \tag{A 10}$$

By introducing the variable Y in the form

$$1-z = \epsilon Y,$$

from (A 10) we obtain

$$p = A2^{\frac{1}{2}D}\epsilon^{2-\frac{1}{2}D}(-\tfrac{1}{4}Y^{2-\frac{1}{2}D}+\omega^2Y^{-\frac{1}{2}D}+2^{-\frac{1}{2}D}\mu^2k_1Y^{\frac{1}{2}D}). \tag{A 11}$$

(Hence the scaling chosen, (A 3), becomes understandable.) We shall call (A 11) the internal asymptotic expansion of the outer solution.

Let us consider now the inner region $1-z = O(\epsilon)$. Assuming $1-z = \epsilon Y$, from (A 1) we get

$$\frac{4}{D^2}Y\frac{\partial}{\partial Y}\frac{Y}{\delta}\frac{\partial P}{\partial Y}-\left\{\frac{1}{\delta}+\frac{2Y}{D\delta^2}\left[-\left(\omega+\frac{Y}{2}\right)+\frac{Y^2(1/D-1)}{\omega-Y/2}\right]\right\}p+O(\epsilon^{4-2D}) = 0. \tag{A 12}$$

Here we designate

$$\delta = (\omega-\tfrac{1}{2}Y)^2-Y^2(1-1/D).$$

We write at leading order the solution of (A 12), bearing in mind its subsequent matching when $Y \to \infty$ with (A 11)

$$p = \{\tilde{A}p_a(Y)+\tilde{B}p_b(Y)\}\epsilon^{2-\frac{1}{2}D}, \tag{A 13}$$

where (cf. (A 2)),
$$p_a(Y) = 2^{\frac{1}{2}D}(\omega^2-\tfrac{1}{4}Y^2)\,Y^{-\frac{1}{2}D}, \tag{A 14}$$

$$p_b(Y) = 2^{-D}p_a(Y)\int_{-\infty}^{Y}\frac{x^{D-1}\,dx}{(\omega^2-\frac{1}{4}x^2)}\left[(\omega-\tfrac{1}{2}x)^2-x^2\left(1-\frac{1}{D}\right)\right].$$

When $Y \to \infty$
$$p_b(Y) \sim k_2Y^{\frac{1}{2}D}, \quad k_2 = -2^{2-D/2}\frac{(1/D-\frac{3}{4})}{D-2}. \tag{A 15}$$

When $Y \to 0$
$$p_b(Y) = 2^{-\frac{1}{2}D}\left[\omega^2Y^{-\frac{1}{2}D}I(\omega,D)+\frac{1}{D}Y^{\frac{1}{2}D}\right], \tag{A 16}$$

where
$$I(\omega,D) = -\int_{0}^{\infty}\frac{x^{D-1}\,dx}{(\omega^2-\frac{1}{4}x^2)}\left[(\omega-\tfrac{1}{2}x)^2-x^2\left(1-\frac{1}{D}\right)\right].$$

Upon substituting (A 14) and (A 16) into (A 13), we obtain for p when $Y \to \infty$

$$p = \{\tilde{A}(\omega^2-\tfrac{1}{4}Y^2)\,Y^{-\frac{1}{2}D}2^{\frac{1}{2}D}+k_2\tilde{B}Y^{\frac{1}{2}D}\}\epsilon^{2-\frac{1}{2}D}. \tag{A 17}$$

The matching of (A 17) and (A 11) gives the connections:

$$\tilde{A} = A, \quad \tilde{B} = \mu^2\frac{k_1}{k_2}A. \tag{A 18}$$

Next, from (A 13), with the aid of (A 14), (A 15) and (A 18), we find that when $Y \to 0$

$$p(Y) = A\left\{2^{\frac{1}{2}D}\omega^2 Y^{-\frac{1}{2}D} + 2^{\frac{1}{2}D}\mu^2\frac{k_1}{k_2}\left(\omega^2 Y^{-\frac{1}{2}D} I + \frac{1}{D} Y^{\frac{1}{2}D}\right)\right\}. \qquad \text{(A 19)}$$

This solution must be matched with the solution in region II (A 4), i.e. with $p \sim Y^{\frac{1}{2}D}$. This yields

$$2^D + \mu^2\frac{k_1}{k_2} I(\omega, D) = 0. \qquad \text{(A 20)}$$

Assuming $\omega = \mathrm{i}\tilde{\gamma}$, we write (A 20) in the form

$$2^D + \mu^2\tilde{\gamma}^{D-2}\frac{k_1}{k_2} J(D) = 0. \qquad \text{(A 21)}$$

By calculating $J(D)$ and k_1/k_2, we get

$$J(D) = -\frac{2^D\pi}{\sin(\pi D)}(1-D)\,\mathrm{e}^{-\mathrm{i}\pi\frac{1}{2}D}, \qquad \text{(A 22)}$$

$$\frac{k_1}{k_2} = \frac{1}{4}\frac{\pi D^2}{\sin(2\pi/D)}(1-2D)(1-\tfrac{2}{3}D)(1-D).$$

With the help of (A 22), from (A 21) we find

$$\tilde{\gamma} = \{\tfrac{1}{8}\mu^2 F(D)\}^{1/(2-D)}\left\{\cos\frac{\pi D}{2(2-D)} - \mathrm{i}\sin\frac{\pi D}{2(2-D)}\right\}. \qquad \text{(A 23)}$$

So that the growth rate and frequency of the $m = 1$ mode are

$$\left.\begin{aligned} \gamma &= \epsilon\,\mathrm{Re}\,\tilde{\gamma} = \{\tfrac{1}{8}M^2 F(D)\}^{1/(2-D)}\cos\frac{\pi D}{2(2-D)}, \\ \Omega_{\mathrm{p}} &= -\epsilon\,\mathrm{Im}\,\tilde{\gamma} = \{\tfrac{1}{8}M^2 F(D)\}^{1/(2-D)}\sin\frac{\pi D}{2(2-D)}. \end{aligned}\right\} \qquad \text{(A 24)}$$

The function $F(D)$ is defined in the main text.

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